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ON THE EXTRACTION OF ROOTS OF WHOLE NUMBERS BY THE “BINOMIAL THEOREM.”

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In many instances the “Binomial Theorem” may be employed with advantage in the extraction of roots of whole numbers.

I.—We have

$$(a^n + x)^{\frac{1}{n}} = a + \frac{x}{n a^{n-1}} - \frac{(n-1)x^2}{1 \cdot 2 \cdot n^2 \cdot a^{2n-1}} + \frac{(n-1)(2n-1)x^3}{1 \cdot 2 \cdot 3 \cdot n^3 \cdot a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot n^4 \cdot a^{4n-1}} + \&c. \dots \dots \dots (1),$$

and

$$(a^n - x)^{\frac{1}{n}} = a - \frac{x}{n a^{n-1}} - \frac{(n-1)x^2}{1 \cdot 2 \cdot n^2 \cdot a^{2n-1}} - \frac{(n-1)(2n-1)x^3}{1 \cdot 2 \cdot 3 \cdot n^3 \cdot a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot n^4 \cdot a^{4n-1}} - \&c. \dots \dots \dots (2).$$

These formulas are very convenient when x is small compared with a .

EXAMPLES.—1. Required the square root of 26. Making $a = 5$, $x = 1$ and $n = 2$, in (1),

$$\sqrt{26} = 5 + \frac{1}{10} - \frac{1}{1000} + \frac{1}{50000} - \frac{1}{2000000} + \&c., \\ = 5.099019 +.$$

2.—Required the square root of 216.

$$216 = 24 \times 9; \therefore \sqrt{216} = 3\sqrt{24}.$$

By (2),

$$\sqrt{24} = 5 - \frac{1}{10} - \frac{1}{1000} - \frac{1}{50000} - \frac{1}{2000000} - \&c., \\ = 4.8989794 +; \\ \therefore \sqrt{216} = 14.696938 +.$$

3.—Required the fifth root of 30.

$$\sqrt[5]{30} = \sqrt[5]{32-2} = 2 - \frac{1}{5 \cdot 2^3} - \frac{1}{5^2 \cdot 2^6} - \frac{3}{5^3 \cdot 2^{10}} - \&c., \\ = 1.974351 +.$$

Also,

$$(a^n + x)^{-\frac{1}{n}} = \frac{1}{a} - \frac{x}{n a^{n+1}} + \frac{(n+1) x^2}{1. 2. n^2. a^{2n+1}} - \frac{(n+1) (2n+1) x^3}{1. 2. 3. n^3. a^{3n+1}} \\ + \frac{(n+1) (2n+1) (3n+1) x^4}{1. 2. 3. 4. n^4. a^{4n+1}} - \&c. \dots (3),$$

and

$$(a^n - x)^{-\frac{1}{n}} = \frac{1}{a} + \frac{x}{n a^{n+1}} + \frac{(n+1) x^2}{1. 2. n^2. a^{2n+1}} + \frac{(n+1) (2n+1) x^3}{1. 2. 3. n^3. a^{3n+1}} \\ + \frac{(n+1) (2n+1) (3n+1) x^4}{1. 2. 3. 4. n^4. a^{4n+1}} + \&c. \dots (4).$$

EXAMPLE.—Required the one-hundredth root of 2.

$$2 = \frac{1}{1 - \frac{1}{2}} = (1 - \frac{1}{2})^{-1}, \therefore (2)^{\frac{1}{100}} = (1 - \frac{1}{2})^{-\frac{1}{100}}.$$

Therefore by (4)

$$(2)^{\frac{1}{100}} = 1 + \frac{1}{200} + \frac{101}{80000} + \frac{6767}{16000000} + \frac{2036867}{12800000000} + \&c., \\ = 1.00695 +, \text{ using 9 terms.}$$

II.—Let a be any number, then

$$a = \frac{a \cancel{p}^n}{\cancel{p}^n} \times \frac{q^n}{q^n} = \frac{\cancel{p}^n}{q^n} \left[1 - \left(\frac{\cancel{p}^n - a q^n}{\cancel{p}^n} \right) \right]; \\ \therefore (a)^{\frac{1}{n}} = \frac{\cancel{p}}{q} \left[1 - \left(\frac{\cancel{p}^n - a q^n}{\cancel{p}^n} \right) \right]^{\frac{1}{n}}, \\ = \frac{\cancel{p}}{q} \left[1 - \left(\frac{\cancel{p}^n - a q^n}{n \cancel{p}^n} \right) - \frac{(n-1)}{1. 2} \left(\frac{\cancel{p}^n - a q^n}{n \cancel{p}^n} \right)^2 \right. \\ \left. - \frac{(n-1) (2n-1)}{1. 2. 3} \left(\frac{\cancel{p}^n - a q^n}{n \cancel{p}^n} \right)^3 - \frac{(n-1) (2n-1) (3n-1)}{1. 2. 3. 4} \left(\frac{\cancel{p}^n - a q^n}{n \cancel{p}^n} \right)^4 - \&c. \right],$$

where \cancel{p} and q may be any numbers chosen at pleasure.

Let r_m be the root of a to m places of decimals, and R_m the remainder; then if $\cancel{p} = (10)^m$ and

$$q = \frac{(10)^m r_m}{a},$$

$$(a)^{\frac{1}{n}} = \frac{(10)^m}{(10)^m \left(\frac{r_m}{a}\right)} \left[1 - \frac{\left(\frac{R_m}{a}\right)}{(10)^{2m}} \right]^{\frac{1}{n}}$$

Take $a = 2$, $n = 2$ and $m = 8$; then

$$\begin{aligned} \sqrt{2} &= \frac{100000000}{70710678} \left[1 - \frac{33560632}{10000000000000000} \right]^{\frac{1}{2}}, \\ &= \frac{100000000}{70710678} \left[1 - \frac{16780316}{10000000000000000} \right. \\ &\quad - \frac{140789502529928}{(10000000000000000)^2} - \frac{2362492341934991297248}{(10000000000000000)^3} \\ &\quad - \frac{4955421005656150678133921296}{(10000000000000000)^4} \\ &\quad \left. - \frac{1164149425431271940117820855324133504}{(10000000000000000)^5} \quad \&c. \right], \\ &= 1.41421356237309504880168872420969807856967187537694 + \end{aligned}$$

Let $a = 3$, $n = 2$ and $m = 13$; then

$$\begin{aligned} \sqrt{3} &= \frac{10000000000000}{5773502691896} \left[1 - \frac{8925087775552}{100000000000000000000000000000000} \right]^{\frac{1}{2}} \\ &= \frac{10000000000000}{5773502691896} \left[1 - \frac{4462543887776}{100000000000000000000000000000000} \right. \\ &\quad - \frac{9957148975163468441113088}{(100000000000000000000000000000000)^2} \\ &\quad \left. - \frac{44434214298790798522333481840396812288}{(100000000000000000000000000000000)^3} \quad \&c. \right], \\ &= 1.73205080756887729352744634150587236694280525381038 + \end{aligned}$$

Four additional terms would give the root true to at least one hundred places of decimals.